MMAT5390 Mathematical Image Processing
Solution of Midterm Examination

1. (a) Let $A = (a_{ij})_{1\le i,j\le 2}$, $B = (b_{ij})_{1\le i,j\le 2}$ and $g = \mathcal{O}(f) \in M_{2\times 2}(\mathbb{R})$, then we have

$$g_{\alpha,\beta} = \sum_{x=1}^{2} a_{\alpha x}(\sum_{y=1}^{2} f(x,y)b_{y\beta}) = \sum_{x=1}^{2}\sum_{y=1}^{2} a_{\alpha x}b_{y\beta}f(x,y),$$

Which means $h^{\alpha,\beta}(x,y) = a_{\alpha x}b_{y\beta}$. Hence the transformation matrix

$$H = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{21} & a_{12}b_{21} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{21} & a_{22}b_{21} \\ a_{11}b_{12} & a_{12}b_{12} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{12} & a_{22}b_{12} & a_{21}b_{22} & a_{22}b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}A & b_{21}A \\ b_{12}A & b_{22}A \end{pmatrix} = B^T \otimes A.$$

(b) We have

$$\begin{aligned} a &= 2\times 2 \\ c+e &= 3\times 2 \\ b+c &= a\times 3 \\ e-c &= 2\times 4 \\ d-a &= 3\times 4 \end{aligned}$$

Hence, we can get $a = 1, b = 4, c = -1, d = 13$ and $e = 7$. Moreover,

$$H = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 4 & 6 \\ 3 & 6 & 4 & 8 \\ 6 & 9 & 8 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = B^T \otimes A$$

Therefore, $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

(c) In order to make h shift-invariant, H is necessary to be block-circulant. i.e.

$$\begin{pmatrix} 4u & u \\ 1 & v \end{pmatrix} = \begin{pmatrix} v & 1 \\ 1 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 2 & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Hence, $u = 1, v = 4$ and $w = 1$.

2. (a) $A^TA = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 25 \end{pmatrix}$

Hence the characteristic polynomial of A^TA is given by

$\det(A^TA - \lambda I_3) = \begin{vmatrix} 10-\lambda & 6 & 0 \\ 0 & 10-\lambda & 0 \\ 0 & 0 & 25-\lambda \end{vmatrix} = -(\lambda - 25)(\lambda - 16)(\lambda - 4).$

Therefore the eighenvalues of A^TA are:

$$\lambda_1 = 25, \quad \lambda_2 = 16, \quad \lambda_3 = 4$$

For $\lambda_1 = 25$,

$$[A^TA - \lambda_1 I_3|0] = \left[\begin{array}{ccc|c} -15 & 6 & 0 & 0 \\ 6 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector, which is also unit.

For $\lambda_2 = 16$,

$$[A^TA - \lambda_2 I_3|0] = \left[\begin{array}{ccc|c} -6 & 6 & 0 & 0 \\ 6 & -6 & 0 & 0 \\ 0 & 0 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector, which gives the unit eigenvector $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = 4$,

$$[A^TA - \lambda_3 I_3|0] = \left[\begin{array}{ccc|c} 6 & 6 & 0 & 0 \\ 6 & 6 & 0 & 0 \\ 0 & 0 & 21 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector, which gives the unit eigenvector $\vec{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Then $\vec{u}_1 = \frac{1}{\sqrt{\lambda_1}} A\vec{v}_1 = \frac{1}{\sqrt{25}} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$

$\vec{u}_2 = \frac{1}{\sqrt{\lambda_2}} A\vec{v}_2 = \frac{1}{\sqrt{16}} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$

$\vec{u}_3 = \frac{1}{\sqrt{\lambda_3}} A\vec{v}_3 = \frac{1}{\sqrt{4}} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$

Hence an svd of A is given by $A = U\Sigma V^T$, where $U = (\vec{u}_1, \vec{u}_2, \vec{u}_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix},$

$\Sigma = \begin{pmatrix} \sqrt{25} & 0 & 0 \\ 0 & \sqrt{16} & 0 \\ 0 & 0 & \sqrt{4} \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $V = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}.$

(b) The elementary images according to the above svd are given by:

$$\vec{u}_1\vec{v}_1^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$\vec{u}_2\vec{v}_2^T = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\vec{u}_3\vec{v}_3^T = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

Hence,

$$\begin{aligned} A &= \sqrt{25}\vec{u}_1\vec{v}_1^T + \sqrt{16}\vec{u}_2\vec{v}_2^T + \sqrt{4}\vec{u}_3\vec{v}_3^T \\ &= 5 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 4 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

(c) Since $A = U\Sigma V^T$, where both U and V are orthogonal matrices, and $\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

We modified Σ to obtain $\tilde{\Sigma} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $\tilde{A} = U\tilde{\Sigma}V^T = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

Obviously, $\|\tilde{A} - A\|_F = \|\tilde{\Sigma} - \Sigma\|_F = 2$ and rank($\tilde{A}$)=rank($\tilde{\Sigma}$)=2

(d) Observe that $N = (\tau_1+\epsilon_1)\vec{u}_1\vec{v}_1^T + \tau_2\vec{u}_2\vec{v}_2^T = (\tau_1+\epsilon_1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \tau_2 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Hence

$$\begin{aligned} \tilde{A} &= (5 + \tau_1 + \epsilon_1)\vec{u}_1\vec{v}_1^T + (4 + \tau_2)\vec{u}_2\vec{v}_2^T + 2\vec{u}_3\vec{v}_3^T \\ &= (5 + \tau_1 + \epsilon_1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (4 + \tau_2) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

3. (a) $\tilde{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$

(b)

$$\begin{aligned}
A_{Haar} &= \tilde{H} A \tilde{H}^T \\
&= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 12 & 0 & -2\sqrt{2} & 2\sqrt{2} \\ 0 & 0 & 0 & 0 \\ -2\sqrt{2} & 0 & 6 & -6 \\ 2\sqrt{2} & 0 & -6 & 6 \end{pmatrix}
\end{aligned}$$

(c)

$$\begin{aligned}
\tilde{A} &= \tilde{H}^T \tilde{A}_{Haar} \tilde{H} \\
&= \tilde{H}^T (A_{Haar} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix})\tilde{H} \\
&= \tilde{H}^T A_{Haar} \tilde{H} + \tilde{H}^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tilde{H} \\
&= A + \frac{1}{4}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= A + \begin{pmatrix} \frac{\epsilon}{2} & -\frac{\epsilon}{2} & 0 & 0 \\ -\frac{\epsilon}{2} & \frac{\epsilon}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Hence, only $\tilde{A}_{1,1}, \tilde{A}_{1,2}, \tilde{A}_{2,1}$ and $\tilde{A}_{2,2}$ are different from A

4. (a) i. Method 1 (directly):

$$\hat{h}(m,n) = \widehat{f \odot g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) g(k,l) e^{-2\pi j \left(\frac{mk}{M} + \frac{nl}{N}\right)},$$

whereas

$$\hat{f} * \hat{g}(m,n) = \sum_{m'=0}^{M-1} \sum_{n'=0}^{N-1} \hat{f}(m',n') \hat{g}(m-m', n-n')$$

$$= \frac{1}{M^2N^2} \sum_{m',k,k'=0}^{M-1} \sum_{n',l,l'=0}^{N-1} f(k,l) e^{-2\pi j \left(\frac{m'k}{M} + \frac{n'l}{N}\right)} g(k',l') e^{-2\pi j \left(\frac{(m-m')k'}{M} + \frac{(n-n')l'}{N}\right)}$$

$$= \frac{1}{M^2N^2} \sum_{m',k,k'=0}^{M-1} \sum_{n',l,l'=0}^{N-1} f(k,l) g(k',l') e^{-2\pi j \left(\frac{mk' + m'(k-k')}{M} + \frac{nl' + n'(l-l')}{N}\right)}$$

$$= \frac{1}{MN} \sum_{k,k'=0}^{M-1} \sum_{l,l'=0}^{N-1} f(k,l) g(k',l') e^{-2\pi j \left(\frac{mk'}{M} + \frac{nl'}{N}\right)} \delta(k-k') \delta(l-l')$$

$$= \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) g(k,l) e^{-2\pi j \left(\frac{mk}{M} + \frac{nl}{N}\right)} = \hat{h}(m,n).$$

ii. Method 2 (iDFT):

$$iDFT(\hat{f} * \hat{g})(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f} * \hat{g}(m,n) e^{2\pi j \left(\frac{mk}{M} + \frac{nl}{N}\right)}$$

$$= \sum_{m,m'=0}^{M-1} \sum_{n,n'=0}^{N-1} \hat{f}(m',n') \hat{g}(m-m', n-n') e^{2\pi j \left(\frac{mk}{M} + \frac{nl}{N}\right)}$$

$$= \frac{1}{M^2N^2} \sum_{m,m',k',k''=0}^{M-1} \sum_{n,n',l',l''=0}^{N-1} f(k',l') g(k'',l'') \quad e^{2\pi j \left(\frac{m(k-k'') + m'(k''-k')}{M} + \frac{n(l-l'') + n'(l''-l')}{N}\right)}$$

$$= \sum_{k',k''=0}^{M-1} \sum_{l',l''=0}^{N-1} f(k',l') g(k'',l'') \mathbf{1}_{M\mathbb{Z}}(k-k'') \mathbf{1}_{M\mathbb{Z}}(k'-k'') \mathbf{1}_{N\mathbb{Z}}(l-l'') \mathbf{1}_{N\mathbb{Z}}(l'-l'')$$

$$= f(k,l) g(k,l) = h(k,l).$$

(b) Assume $g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$, then $f \odot g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3g_{21} & 0 & 0 & 0 \\ g_{31} & 0 & 0 & 0 \\ 3g_{41} & 0 & 0 & 0 \end{pmatrix}$,

$$\widehat{f \odot g} = U(f \odot g)U = \tfrac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3g_{21} & 0 & 0 & 0 \\ g_{31} & 0 & 0 & 0 \\ 3g_{41} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$= \tfrac{1}{16} \begin{pmatrix} 3g_{21} + g_{31} + 3g_{41} & 3g_{21} + g_{31} + 3g_{41} & 3g_{21} + g_{31} + 3g_{41} & ... \\ -g_{31} + 3(g_{41} - g_{21})j & -g_{31} + 3(g_{41} - g_{21})j & -g_{31} + 3(g_{41} - g_{21})j & ... \\ g_{31} - 3g_{21} - 3g_{41} & g_{31} - 3g_{21} - 3g_{41} & g_{31} - 3g_{21} - 3g_{41} & ... \\ -g_{31} + 3(g_{21} - g_{41})j & -g_{31} + 3(g_{21} - g_{41})j & -g_{31} + 3(g_{21} - g_{41})j & ... \end{pmatrix}$$

$$= \hat{f} * \hat{g} = \begin{pmatrix} 13 & 13 & 13 & 13 \\ -1 & -1 & -1 & -1 \\ -11 & -11 & -11 & -11 \\ -1 & -1 & -1 & -1 \end{pmatrix}$$

By elementwise comparison, we must have

$$\begin{aligned} 3g_{21} + g_{31} + 3g_{41} &= 13 \times 16 \\ -g_{31} + 3(g_{41} - g_{21})j &= -1 \times 16 \\ g_{31} - 3g_{21} - 3g_{41} &= -11 \times 16 \\ -g_{31} + 3(g_{21} - g_{41})j &= -1 \times 16 \end{aligned}$$

Therefore we know that $g = \begin{pmatrix} * & * & * & * \\ 32 & * & * & * \\ 16 & * & * & * \\ 32 & * & * & * \end{pmatrix}$, where $*$ stands for any number.

5. $\tilde{g}(k,l) = g(-k-a_0, l-b_0)$ for any $0 \le k \le N-1, 0 \le l \le N-1$. Then

$$
\begin{aligned}
\hat{\tilde{g}}(m,n) &= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\tilde{g}(k,l)e^{-2\pi j\frac{mk+nl}{N}} \\
&= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}g(-k-a_0,l-b_0)e^{-2\pi j\frac{mk+nl}{N}} \\
(let \quad k'=-k-a_0, l'=l-b_0) &= \frac{1}{N^2}\sum_{k'=1-N-a_0}^{-a_0}\sum_{l'=-b_0}^{N-1-b_0}g(k',l')e^{-2\pi j\frac{-m(k'+a_0)+n(l'+b_0)}{N}} \\
&= \frac{1}{N^2}\sum_{k'=1-N-a_0}^{-a_0}\sum_{l'=-b_0}^{N-1-b_0}g(k',l')e^{-2\pi j\frac{-mk'+nl'}{N}}e^{-2\pi j\frac{-a_0m+b_0n}{N}} \\
&= \frac{1}{N^2}e^{-2\pi j\frac{-a_0m+b_0n}{N}}\sum_{k'=1-N-a_0}^{-a_0}\sum_{l'=-b_0}^{N-1-b_0}g(k',l')e^{-2\pi j\frac{-mk'+nl'}{N}} \\
&= \frac{1}{N^2}e^{-2\pi j\frac{-a_0m+b_0n}{N}}\sum_{k'=1-N-a_0}^{-a_0}(\sum_{l'=0}^{N-1-b_0}+\sum_{l'=-b_0}^{-1}g(k',l')e^{-2\pi j\frac{-mk'+nl'}{N}}) \\
(\text{letting } l''=l'+N) &= \frac{1}{N^2}e^{-2\pi j\frac{-a_0m+b_0n}{N}}\sum_{k'=1-N-a_0}^{-a_0}(\sum_{l'=0}^{N-1-b_0}g(k',l')e^{-2\pi j\frac{-mk'+nl'}{N}} \\
&+ \sum_{l''=N-b_0}^{N-1}g(k',l''-N)e^{-2\pi j\frac{-mk'+n(l''-N)}{N}}) \\
(\text{by periodicity and } e^{2\pi jn}=1) &= \frac{1}{N^2}e^{-2\pi j\frac{-a_0m+b_0n}{N}}\sum_{k'=1-N-a_0}^{-a_0}(\sum_{l'=0}^{N-1-b_0}g(k',l')e^{-2\pi j\frac{-mk'+nl'}{N}} \\
&+ \sum_{l''=N-b_0}^{N-1}g(k',l'')e^{-2\pi j\frac{-mk'+nl''}{N}}) \\
(\text{rewrite } l' \text{ and } l'' \text{ as } l) &= \frac{1}{N^2}e^{-2\pi j\frac{-a_0m+b_0n}{N}}\sum_{k'=1-N-a_0}^{-a_0}\sum_{l=0}^{N-1}g(k',l)e^{-2\pi j\frac{-mk'+nl}{N}} \\
(\text{letting } k''=k'+N) &= \frac{1}{N^2}e^{-2\pi j\frac{-a_0m+b_0n}{N}}\sum_{l=0}^{N-1}(\sum_{k''=1-a_0}^{N-a_0}g(k''-N,l)e^{-2\pi j\frac{-m(k''-N)+nl}{N}}) \\
(\text{by periodicity and } e^{2\pi jm}=1) &= \frac{1}{N^2}e^{-2\pi j\frac{-a_0m+b_0n}{N}}\sum_{l=0}^{N-1}(\sum_{k''=1-a_0}^{N-a_0}g(k'',l)e^{-2\pi j\frac{-m(k'')+nl}{N}}) \\
&= \frac{1}{N^2}e^{-2\pi j\frac{-a_0m+b_0n}{N}}\sum_{l=0}^{N-1}(\sum_{k''=1-a_0}^{-1} \\
&+ \sum_{k''=0}^{N-a_0}g(k'',l)e^{-2\pi j\frac{-m(k'')+nl}{N}}) \\
(\text{letting } k'''=k''+N) &= \frac{1}{N^2}e^{-2\pi j\frac{-a_0m+b_0n}{N}}\sum_{l=0}^{N-1}(\sum_{k'''=N-a_0+1}^{N-1}g(k'''-N,l)e^{-2\pi j\frac{-m(k'''-N)+nl}{N}} \\
&+ \sum_{k''=0}^{N-a_0}g(k'',l)e^{-2\pi j\frac{-m(k'')+nl}{N}})
\end{aligned}
$$

$$
\begin{aligned}
\text{(by periodicity and } e^{2\pi jm}=1) &= \frac{1}{N^2} e^{-2\pi j\frac{-a_0 m+b_0 n}{N}} \sum_{l=0}^{N-1} \Big(\sum_{k'''=N-a_0+1}^{N-1} g(k''',l) e^{-2\pi j\frac{-mk'''+nl}{N}} \\
&\quad + \sum_{k''=0}^{N-a_0} g(k'',l) e^{-2\pi j\frac{-m(k'')+nl}{N}} \Big) \\
\text{(rewrite } k'' \text{ and } k''' \text{ as } k) &= \frac{1}{N^2} e^{-2\pi j\frac{-a_0 m+b_0 n}{N}} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} g(k'',l) e^{-2\pi j\frac{-mk+nl}{N}} \\
&= \frac{1}{N^2} e^{-2\pi j\frac{-a_0 m+b_0 n}{N}} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k'',l) e^{-2\pi j\frac{-mk+nl}{N}} \\
&= e^{-2\pi j\frac{-a_0 m+b_0 n}{N}} \hat{g}(-m,n).
\end{aligned}
$$